

# Duality of certain Banach spaces of vector-valued holomorphic functions

F. J. Bertoloto\*

Faculdade de Matemática  
Universidade Federal de Uberlândia  
38.400-902 - Uberlândia - Brazil  
e-mail: bertoloto@famat.ufu.br

## Abstract

In this work we study the vector-valued Hardy spaces  $H^p(\mathbb{D}; F)$  ( $1 \leq p \leq \infty$ ) and their relationship with RNP, ARNP and the UMDP properties. By following the approach of Taylor [24],[25] in the scalar-valued case, we prove that, when  $F$  and  $F'$  have the ARNP property, then  $H^p(\mathbb{D}; F)'$  and  $H^q(\mathbb{D}; F')$  are canonically topologically isomorphic (for  $p, q \in (1, \infty)$  conjugate indices) if and only if  $F$  has the UMDP.

## 1 Introduction

Throughout this paper let  $\mathbb{D} = \{z \in \mathbb{C}; |z| < 1\}$ ,  $\mathbb{T} = \{z \in \mathbb{C}; |z| = 1\}$ , and let  $F$  be a complex Banach space. Let  $\mathcal{H}(\mathbb{D}; F)$  denote the vector space of all holomorphic functions  $f: \mathbb{D} \rightarrow F$ . Let  $1 \leq p, q \leq \infty$  with  $\frac{1}{p} + \frac{1}{q} = 1$ .

This paper is devoted to the study of subspaces of  $\mathcal{H}(\mathbb{D}; F)$  that satisfy certain axioms. Our paper is indeed a vector-valued versions of Taylor [24, 25]. Our main objective is the study of the vector-valued Hardy space  $H^p(\mathbb{D}; F)$ . In particular we show that if  $F$  and  $F'$  have the analytic Radon-Nikodym property (ARNP) and  $1 < p < \infty$ , then the spaces  $H^p(\mathbb{D}; F)'$  and  $H^q(\mathbb{D}; F')$  are canonically topologically isomorphic if and only if  $F$  has the unconditional martingale difference property (UMDP).

This paper is organized as follows. Section 2 is devoted to the study of subspaces of  $\mathcal{H}(\mathbb{D}; F)$  satisfying certain axioms. In particular we study the spaces  $H^p(\mathbb{D}; F)$ . Section 3 is devoted to the study of the space  $H^p(\mathbb{T}; F)$  of boundary-value functions associated with  $H^p(\mathbb{D}; F)$ . There

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we introduce the ARNP following Bukhvalov [5]. Section 4 is devoted to the study of duality theory for the spaces  $H^p(\mathbb{D}; F)$ , and its connection with the UMDP. Finally Section 5 is devoted to the study of the spaces  $H^p(\mathbb{D}; F)_w$  and  $L^p(\mathbb{T}; F)_w$ , the weak versions of the spaces  $H^p(\mathbb{D}; F)$  and  $L^p(\mathbb{T}; F)$ .

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## 2 Preliminaries

For a few elementary facts regarding the Bochner integral we refer to Mujica [14]. Except for Proposition 2.16, that improves a result of Taylor [24, Theorem 8.1], all the other results have proofs similar to those given by Taylor [24, 25] in the scalar-valued case. We cite Lemma 2.6 with a proof in order to highlight its importance for other results. Here, we change the notation of Taylor trying to adapt to a more current one.

### 2.1 Vector-valued holomorphic functions on the open unit disc

It is well known that if  $f \in \mathcal{H}(\mathbb{D}; F)$  then

$$f(z) = \sum_{n=0}^{\infty} \gamma_n(f) z^n,$$

where  $\gamma_n(f) \in F$ . This series converges uniformly and absolutely in the open disks  $D_r = \{z; |z| < r\}$  for  $0 < r < 1$ . It is easy to see that

$$\gamma_n(\alpha f + \beta g) = \alpha \gamma_n(f) + \beta \gamma_n(g),$$

for every  $f, g \in \mathcal{H}(\mathbb{D}; F)$  and  $\alpha, \beta \in \mathbb{C}$ .

**Definition 2.1.** Let  $z \in \mathbb{D}$ .

- a) If  $v \in F$  and  $\|v\| = 1$ , then we define  $u_n^v(z) = z^n v$ .
- b) If  $f \in \mathcal{H}(\mathbb{D}; F)$  and  $t \in \mathbb{R}$ , then we define  $U_t f(z) = f(z e^{it})$ .
- c) If  $f \in \mathcal{H}(\mathbb{D}; F)$  and  $|w| \leq 1$ , then we write  $T_w f(z) = f(zw)$ .

The reader can verify that  $U_t = T_{e^{it}}$  and  $\gamma_n(T_w f) = w^n \gamma_n(f)$  for every  $t \in \mathbb{R}$ ,  $w \in \mathbb{D}$  and  $n \in \mathbb{N}$ . Besides,  $U_t$  and  $T_w$  are linear operators on  $\mathcal{H}(\mathbb{D}; F)$ .

**Definition 2.2.** Let  $f \in \mathcal{H}(\mathbb{D}; F)$  and  $g \in \mathcal{H}(\mathbb{D}; F')$ . Then:

$$f(z) = \sum_{n=0}^{\infty} \gamma_n(f) z^n \quad \text{and} \quad g(z) = \sum_{n=0}^{\infty} \gamma_n(g) z^n.$$

For each  $z \in \mathbb{D}$ , let

$$B(f, g; z) = \sum_{n=0}^{\infty} \langle \gamma_n(f), \gamma_n(g) \rangle z^n,$$

where  $\langle \gamma_n(f), \gamma_n(g) \rangle = \gamma_n(g)(\gamma_n(f))$ .

**Proposition 2.3.** Let  $f \in \mathcal{H}(\mathbb{D}; F)$ ,  $g \in \mathcal{H}(\mathbb{D}; F')$ ,  $w \in \mathbb{D}$  and  $z \in \mathbb{D}$ . Then,

- a)  $B(f, g; \cdot) \in \mathcal{H}(\mathbb{D})$ .
- b)  $B(f, g; z)$  is linear on  $f$  for each  $g$  and  $z$ .  
 $B(f, g; z)$  is linear on  $g$  for each  $f$  and  $z$ .
- c)  $B(T_w f, g; z) = B(f, g, wz)$ .
- d)  $B(f, g; z) = \frac{1}{2\pi} \int_0^{2\pi} \langle f(z_1 e^{i\theta}), g(z_2 e^{-i\theta}) \rangle d\theta$  whenever  $z = z_1 z_2$ , with  $z_1, z_2 \in \mathbb{D}$ .

Let  $H \subset \mathcal{H}(\mathbb{D}; F)$  be a complex normed vector space. We say  $H$  is a *normed space of type  $\mathcal{H}(\mathbb{D}; F)$* , or is of *type  $\mathcal{H}(\mathbb{D}; F)$* , if it contains at least two elements. If  $H$  is a Banach space, we say  $H$  is a *Banach space of type  $\mathcal{H}(\mathbb{D}; F)$* .

Let  $A > 0$  and  $n \in \mathbb{N}$ . We introduce now seven properties (or axioms) that a space  $H$  of type  $\mathcal{H}(\mathbb{D}; F)$  can satisfy.

$P_1$ : There exists a constant  $A$  such that  $\|\gamma_n(f)\| \leq A\|f\|$  if  $f \in H$ . The least such  $A$  we denote by  $A_1(H)$ . Thus each  $\gamma_n: H \rightarrow F$  is a continuous linear operator and  $\|\gamma_n\| \leq A_1(H)$ .

$P_2$ :  $u_n^v \in H$  and there exists a constant  $A$  such that  $\|u_n^v\| \leq A$  for all  $v \in F$  with  $\|v\| = 1$ . The least such  $A$  we denote by  $A_2(H)$ . Thus  $\|u_n^v\| \leq A_2(H)$ .

For the axioms  $P_3$  and  $P_4$ , we need Definition 2.1.

$P_3$ :  $U_t f \in H$  if  $f \in H$  and  $t \in \mathbb{R}$ . Besides  $\|U_t f\| = \|f\|$  and  $U_t$  is an isometry.

$P_4$ : If  $f \in H$  and  $0 < r < 1$ , then  $T_r f \in H$ . There exists a constant  $A$  such that  $\|T_r f\| \leq A\|f\|$ . The least such  $A$  we denote by  $A_4(H)$ . Thus  $\|T_r\| \leq A_4(H)$ .

$P_5$ : If  $f \in H$  and  $0 < r < 1$ , then  $T_r f \in H$  and  $\|f\| = \sup_{0 \leq r < 1} \|T_r f\|$ .

$P_6$ : If  $f \in H$  and  $0 < r < 1$ , then  $T_r f \in H$  and  $\lim_{r \rightarrow 1} \|T_r f - f\| = 0$ .

$P_7$ : If  $f \in \mathcal{H}(\mathbb{D}; F)$  is such that  $T_r f \in H$  whenever  $0 < r < 1$  and  $\sup_{0 \leq r < 1} \|T_r f\| < \infty$ , then  $f \in H$  and  $\|f\| = \sup_{0 \leq r < 1} \|T_r f\|$ .

We write  $H'$  for the topological dual of  $H$ . We can express the constants  $A_1(H)$ ,  $A_2(H)$  and  $A_4(H)$  as follows:

$$A_1(H) = \sup_{n \in \mathbb{N}} \|\gamma_n\|, \quad A_2(H) = \sup_{\substack{v \in F, \|v\|=1 \\ n \in \mathbb{N}}} \|u_n^v\|, \quad A_4(H) = \sup_{0 \leq r < 1} \|T_r\|.$$

**Proposition 2.4.** Let  $H$  be of type  $\mathcal{H}(\mathbb{D}; F)$ . Then

- a) If  $H$  satisfies  $P_1$  e  $P_2$ , then  $A_1(H)A_2(H) \geq 1$ .
- b) If  $H$  satisfies  $P_4$  and for some  $n \in \mathbb{N}$ ,  $u_n \in H$  (in particular, if  $H$  satisfies  $P_2$ ), then  $A_4(H) \geq 1$ .
- c)  $P_5$  implies  $P_4$  and  $A_4(H) = 1$ .
- d)  $P_4$  and  $P_7$  imply  $P_5$ .

**Proof:** For assertion (a) we observe that  $\gamma_n(u_n^v) = v$ . Then if the properties  $P_1$  and  $P_2$  are satisfied, it follows that:

$$1 = \|\gamma_n(u_n^v)\| \leq A_1(H)A_2(H).$$

For assertion (b), we observe that  $T_r(u_n^v) = r^n u_n^v$ . Assertions (c) and (d) are not difficult to prove. ■

A given normed space  $H$  is said to be a *normed space of type  $\mathcal{H}(\mathbb{D}; F)_k$* , if it is a normed space of type  $\mathcal{H}(\mathbb{D}; F)$  that satisfies  $P_1, \dots, P_k$  ( $k = 1, \dots, 7$ ). We say *Banach space of type  $\mathcal{H}(\mathbb{D}; F)_k$*  if  $H$  is a Banach space.

**Definition 2.5.** Let  $H^p(\mathbb{D}; F)$  denote the space of all  $f \in \mathcal{H}(\mathbb{D}; F)$  such that  $\|f\|_p < \infty$ , where

$$\|f\|_p = \sup_{0 \leq r < 1} \left( \frac{1}{2\pi} \int_0^{2\pi} \|f(re^{i\theta})\|^p d\theta \right)^{\frac{1}{p}} \text{ if } 1 \leq p < \infty$$

and

$$\|f\|_\infty = \sup_{z \in \mathbb{D}} \|f(z)\|.$$

Then  $H^p(\mathbb{D}; F)$  is a Banach space for the norm  $\|\cdot\|_p$ . The proof of Rudin [21, page 331] in the scalar-valued case applies. When  $F = \mathbb{C}$ , we write  $H^p(\mathbb{D})$  instead of  $H^p(\mathbb{D}; \mathbb{C})$ .

Next we will see that the spaces  $H^p(\mathbb{D}; F)$  are examples of Banach spaces of type  $\mathcal{H}(\mathbb{D}; F)_4$ . To see this we need the following lemma.

**Lemma 2.6.** For each  $0 \leq r < 1$ ,  $1 \leq p \leq \infty$  and  $f \in \mathcal{H}(\mathbb{D}; F)$ , let

$$\|f\|_{p,r} = \left( \frac{1}{2\pi} \int_0^{2\pi} \|f(re^{i\theta})\|^p d\theta \right)^{\frac{1}{p}}$$

and

$$\|f\|_{\infty,r} = \max_{0 \leq \theta \leq 2\pi} \|f(re^{i\theta})\|.$$

Then the maps  $r \mapsto \|f\|_{p,r}$  and  $p \mapsto \|f\|_{p,r}$  are nondecreasing functions of  $r$  and  $p$  respectively. Also,

$$\lim_{p \rightarrow \infty} \|f\|_{p,r} = \|f\|_{\infty,r}, \quad \lim_{r \rightarrow 1} \|f\|_{p,r} = \|f\|_p. \quad (2.1)$$

**Proof:** Let  $1 \leq p < p_1$ . We consider the functions

$$\phi = \|f\|^p \in L^{\frac{p_1}{p}}(T) \quad \text{and} \quad \psi = 1 \in L^{\frac{p_1}{p_1-p}}. \quad (2.2)$$

Now is enough to apply the Holder's inequality.

The Holder's inequality also resolves the case for  $r$ , when  $1 \leq p < \infty$ , with the help of the Cauchy's Integral Formula (see, e.g., [11, page 8]). When  $p = \infty$  we use the Maximum Modulus Theorem (see, e.g., Mujica [14] or Thorp [26, page 641]).

Concerning the first equality in (2.1), we observe that the limit exists, since  $p \mapsto \|f\|_{p,r}$  is a non-decreasing function of  $p$  that is bounded by  $\|f\|_{\infty,r}$ .

Given  $\epsilon > 0$ , we can assure there exists a set  $A \subset [0, 2\pi]$  of positive Lebesgue measure  $\rho \leq 1$  satisfying,

$$\|f(re^{i\theta})\| > \|f\|_{\infty,r} - \epsilon$$

for every  $\theta \in A$ . Hence,

$$\|f\|_{p,r} \geq \left( \frac{1}{2\pi} \int_A \|f(re^{i\theta})\|^p d\theta \right)^{\frac{1}{p}} \geq (\|f\|_{\infty,r} - \epsilon)(\rho)^{\frac{1}{p}}.$$

Taking  $p \rightarrow \infty$ , the lemma is proved, since the second equality in (2.1) is clear. ■

**Proposition 2.7.** If  $1 \leq p \leq \infty$ , then  $H^p(\mathbb{D}, F)$  is a Banach space of type  $\mathcal{H}(\mathbb{D}; F)_4$ . Also,  $A_k(H^p(\mathbb{D}, F)) = 1$  ( $k = 1, 2, 4$ ).

**Proof:** By Lemma 2.6 and the proof in the scalar-valued case, the proposition follows. See Taylor [25, Theorem 11.1]. ■

**Remark 2.8.** The following equalities hold true for  $f \in \mathcal{H}(\mathbb{D}; F)$  and  $\rho > 0$  such that  $0 \leq r < \rho < 1$ :

$$\|T_r f\|_{p,\rho} = \left( \frac{1}{2\pi} \int_0^{2\pi} \|T_r f(\rho e^{i\theta})\|^p d\theta \right)^{\frac{1}{p}} = \left( \frac{1}{2\pi} \int_0^{2\pi} \|f(r\rho e^{i\theta})\|^p d\theta \right)^{\frac{1}{p}} = \|f\|_{p,r\rho}. \quad (2.3)$$

Thus, from equation (2.3), for each  $f \in \mathcal{H}(\mathbb{D}; F)$  we have

$$\|T_r f\|_p = \|f\|_{p,r}. \quad (2.4)$$

This happens since  $r \mapsto \|f\|_{p,r}$  is a nondecreasing function of  $r$  (Lemma 2.6).

**Proposition 2.9.** The space  $H^\infty(\mathbb{D}, F)$  is a Banach space of type  $\mathcal{H}(\mathbb{D}; F)_5$ . It also satisfies  $P_7$  but not  $P_6$ .

**Proof:** The proof in the scalar-valued case applies. See Taylor [25, Theorem 12.1]. ■

## 2.2 Additional Properties of the normed spaces of type $\mathcal{H}(\mathbb{D}; F)_k$

**Proposition 2.10.** If  $H$  is a normed space of type  $\mathcal{H}(\mathbb{D}; F)_1$ ,  $f \in H$  and  $z \in \mathbb{D}$ , then:

$$\|f(z)\| \leq \frac{A_1(H)\|f\|}{1 - |z|}.$$

**Proof:** By the expansion  $f(z) = \sum_{n=0}^{\infty} \gamma_n(f)z^n$ , the result follows. ■

**Definition 2.11.** Let  $H$  be of type  $\mathcal{H}(\mathbb{D}; F)_1$ ,  $f \in H$ ,  $g \in \mathcal{H}(\mathbb{D}, F')$  and  $z \in \mathbb{D}$ . Then

$$N(g; z) = \sup_{\|f\|=1} |B(f, g; z)|.$$

$N(g; z)$  has the following properties:

- a)  $N(g + h; z) \leq N(g; z) + N(h; z);$
- b)  $N(ag; z) = |a|N(g; z);$
- c)  $N(T_w g; z) = N(g; wz), |w| \leq 1.$
- d)  $N(g; z) = N(g; |z|).$
- e) If  $N(g; r) = N(g; z)$ , where  $|z| = r$ , then  $N(g; r)$  is a continuous and increasing function of  $r$ .
- f)  $N(g; r) = 0$ , for  $0 \leq r < 1$ ,

The proof of properties (a), (b) and (c) is easy. We omit it. For item (d), (e) and (f), see Taylor [24, Theorem 6.1].

**Proposition 2.12.** Let  $H$  be a Banach space of type  $\mathcal{H}(\mathbb{D}; F)_2$ ,  $f \in \mathcal{H}(\mathbb{D}; F)$  and  $w \in \mathbb{D}$ . Then  $T_w f \in H$  and the function  $w \mapsto T_w f$  is analytic on  $\mathbb{D}$  with series expansion:

$$T_w f = \sum_{n=0}^{\infty} a_n \cdot u_n^{v_n}, \quad (2.5)$$

where

- a)  $a_n = \|\gamma_n(f)\|w^n$  e  $v_n = \frac{\gamma_n(f)}{\|\gamma_n(f)\|}$ , if  $\gamma_n(f) \neq 0$ ;
- b)  $a_n = 0 \in \mathbb{C}$  e  $v_n = 0 \in F$ , if  $\gamma_n(f) = 0$ .

**Proof:** The proof in the scalar-valued case applies. See Taylor [24, Theorem 4.1]. ■

## 2.3 Holomorphic functions with values in a dual Banach space

In this section we will always consider spaces  $H$  of type  $\mathcal{H}(\mathbb{D}; F)_3$ , and from them, we will define a space of type  $\mathcal{H}(\mathbb{D}; F')_4$  that we call  $H^b$ .

**Definition 2.13.** Let  $H$  be of type  $\mathcal{H}(\mathbb{D}; F)_3$ . Define  $H^b$  as the set of  $g \in \mathcal{H}(\mathbb{D}, F')$  such that  $N(g; r)$  is a bounded function of  $r$ . We write for  $g \in H^b$ ,

$$N(g) = \lim_{r \rightarrow 1} N(g; r)$$

where the limit can be replaced by  $N(g) = \sup_{0 \leq r < 1} N(g; r)$  from property (e) after Definition 2.11.

**Proposition 2.14.** If  $H$  is of type  $\mathcal{H}(\mathbb{D}; F)_3$ , then  $H^b$  is a Banach space with norm  $N(g)$ .

**Proof:** The proof in the scalar-valued case applies. See Taylor [24, Theorem 7.1]. ■

**Notation:** From now on  $N(g)$  will be denoted by  $\|g\|^b$ , for all  $g \in H^b$ .

**Proposition 2.15.** Let  $H$  be of type  $\mathcal{H}(\mathbb{D}; F)_3$ . Then  $H^b$  is a Banach space of type  $\mathcal{H}(\mathbb{D}; F')_4$ . The constants  $A_k(H^b)$ ,  $k = 1, 2, 4$ , satisfy

- a)  $A_1(H^b) \leq A_2(H)$ .
- b)  $A_2(H^b) \leq A_1(H)$ .
- c)  $A_4(H^b) = 1$ .

**Proof:** See the scalar-valued case in Taylor [24, Theorem 7.2]. The proof here is similar. ■

**Proposition 2.16.** Let  $H$  be a space of type  $\mathcal{H}(\mathbb{D}; F)_4$ . Let  $\gamma \in H'$  and let  $g: \mathbb{D} \rightarrow F'$  be defined by

$$g(z) = \sum_{n=0}^{\infty} z^n \gamma_n \quad (2.6)$$

for all  $z \in \mathbb{D}$ , where  $\gamma_n(v) := \gamma(u_n^v)$  for every  $v \in F$ . Then  $g \in H^b$  and  $\|g\|^b \leq A_4(H)\|\gamma\|$ .

**Proof:** Firstly, we have  $v \neq 0$

$$\begin{aligned} |\gamma_n(v)| = |\gamma(u_n^v)| &\leq \|\gamma\| \|u_n^v\| \\ &= \|\gamma\| \cdot \|v\| \cdot \|u_n^{\frac{v}{\|v\|}}\| \\ &\leq \|\gamma\| A_2(H) \|v\|. \end{aligned}$$

Thus,  $\|\gamma_n\| \leq \|\gamma\| A_2(H)$  which implies  $\gamma_n \in F'$  and  $g \in \mathcal{H}(\mathbb{D}; F')$ . By Proposition 2.12, we have for  $w \in \mathbb{D}$  and  $f \in H$  that

$$T_w f = \sum_{n=0}^{\infty} a_n \cdot u_n^{v_n},$$

is a element of  $H$ , where

- i)  $a_n = \|\gamma_n(f)\| w^n$  and  $v_n = \frac{\gamma_n(f)}{\|\gamma_n(f)\|}$ , if  $\gamma_n(f) \neq 0$ .
- ii)  $a_n = 0 \in \mathbb{C}$ , if  $\gamma_n(f) = 0$ .

Let  $\gamma \in H'$  and  $\{a_{n_k}\}_{k \in \mathbb{N}}$  the subset of nonzero elements of  $\{a_n\}_{n \in \mathbb{N}}$ . Then,

$$\gamma(T_w f) = \sum_{k=0}^{\infty} a_{n_k} \gamma(u_{n_k}^{v_{n_k}}) = \sum_{k=0}^{\infty} \|\gamma_{n_k}(f)\| w^{n_k} \gamma(u_{n_k}^{v_{n_k}}) = \sum_{k=0}^{\infty} \gamma(u_{n_k}^{\gamma_{n_k}(f)}) w^{n_k} = \sum_{k=0}^{\infty} \gamma_{n_k}(\gamma_{n_k}(f)) w^{n_k}.$$

where the last sum comes from definition of  $\gamma$  and is precisely  $B(f, g; w)$ , that is

$$\gamma(T_w f) = B(f, g; w) \quad (2.7)$$

and so, for  $0 \leq r < 1$  we obtain  $|B(f, g; r)| \leq \|\gamma\| \|T_r f\| \leq \|\gamma\| A_4(H) \|f\|$ . Then  $g \in H^b$  and  $\|g\|^b \leq A_4(H) \|\gamma\|$ , completing the proof. ■

**Definition 2.17.** Let  $H$  be a Banach space of type  $\mathcal{H}(\mathbb{D}; F)_4$ . With the notation of Proposition 2.16 let  $\Gamma: H' \rightarrow H^b$  be defined by  $\Gamma(\gamma) = g$ . The mapping  $\Gamma$  is clearly linear.

**Proposition 2.18.** Let  $H$  be a Banach space of type  $\mathcal{H}(\mathbb{D}; F)_4$  satisfying  $P_6$ . Then  $\Gamma: H' \rightarrow H^b$  is a isometric isomorphism.

**Proof:** The proof in the scalar-valued case applies. See Taylor [24, Theorem 9.3]. ■

The next result has a great importance when we study duality of Hardy spaces:

**Theorem 2.19.** Let  $H$  be a Banach space of type  $\mathcal{H}(\mathbb{D}; F)_4$  satisfying that for all  $f \in H$

$$\lim_{r \rightarrow 1} \|T_r f - f\| = 0.$$

Then, every  $\gamma \in H'$  can be represented as follows:

$$\gamma(f) = \lim_{r \rightarrow 1} \frac{1}{2\pi} \int_0^{2\pi} \langle f(\rho e^{i\theta}), g\left(\left(\frac{r}{\rho}\right) e^{-i\theta}\right) \rangle d\theta, \quad (2.8)$$

where  $0 \leq r < \rho < 1$  and  $g \in H^b$ . The element  $g \in H^b$  uniquely determines and is uniquely determined by  $\gamma$  and  $\|\gamma\| = \|g\|^b$ .

**Proof:** The proof in the scalar-valued case applies. See Taylor [24, Theorem 10.1]. ■

**Remark 2.20.** From now on, if  $H = H^p(\mathbb{D}; F)$ , we assign, for each  $g \in \mathcal{H}(\mathbb{D}; F')$ ,

$$N_q(g; r) = \sup_{\|f\|_p=1} |B(f, g; r)|$$

instead of  $N(g; z)$  and, if  $g \in H^p(\mathbb{D}, F)^b$ , we write

$$\|g\|_q^b = N_q(g) = \sup_{0 \leq r < 1} N_q(g; r).$$

**Proposition 2.21.** Let  $1 \leq p \leq \infty$ . If  $g \in \mathcal{H}(\mathbb{D}; F')$ , with  $q$  being the conjugate indice of  $p$ , we have

$$N_q(g; r) \leq \|g\|_{q,r} \quad (2.9)$$

which implies  $H^q(\mathbb{D}; F') \subset H^p(\mathbb{D}; F)^b$ .

**Proof:** The proof in the scalar-valued case applies. See Taylor [25, Theorem 16.2]. ■



### 3 Boundary values and the analytic Radon-Nikodym property

**Definition 3.1.** Let  $1 \leq p \leq \infty$ , and let  $H^p(\mathbb{T}; F)$  be defined by

$$H^p(\mathbb{T}; F) = \{f \in L^p(\mathbb{T}; F); \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) e^{-in\theta} d\theta = 0, n = -1, -2, \dots\}.$$

Clearly  $H^p(\mathbb{T}; F)$  is a closed subspace of  $L^p(\mathbb{T}; F)$ .

**Definition 3.2.** Let  $\varphi \in L^1(\mathbb{T}; F)$  and  $f$  be defined by

$$f(z) = \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{\varphi(\xi)}{\xi - z} d\xi, |z| < 1. \quad (3.1)$$

Of course  $f \in H^p(\mathbb{D}; F)$ . We say  $f$  is the Cauchy integral of  $\varphi$ .

The following result of Ryan extends a classical theorem of Riesz in the case of scalar-valued functions.

**Theorem 3.3. (Ryan [22, 23])** Let  $1 \leq p \leq \infty$ , and let  $F$  be a separable and reflexive Banach space. For each  $\varphi \in H^p(\mathbb{T}; F)$  let  $f: \mathbb{D} \rightarrow F$  be defined by the Cauchy integral (3.1). Then the mapping  $\varphi \rightarrow f$  is an isometric isomorphism between  $H^p(\mathbb{T}; F)$  and  $H^p(\mathbb{D}; F)$ . Moreover the limit

$$\varphi(e^{i\theta}) = \lim_{r \rightarrow 1} f(re^{i\theta})$$

exists almost everywhere in the norm of  $F$

Ryan's result was improved by Bukhvalov as follows.

**Theorem 3.4. (Bukhvalov [5])** Let  $1 \leq p \leq \infty$ . For each  $\varphi \in H^p(\mathbb{T}; F)$  let  $f: \mathbb{D} \rightarrow F$  be defined by the Cauchy integral (3.1). Then the mapping  $\varphi \mapsto f$  is an isometric isomorphism between  $H^p(\mathbb{T}; F)$  and the closed subspace of all  $f \in H^p(\mathbb{D}; F)$  such that the radial limits

$$\varphi(e^{i\theta}) = \lim_{r \rightarrow 1} f(re^{i\theta}) \quad (3.2)$$

exist almost everywhere in the norm of  $F$ . If  $\varphi_r(e^{i\theta}) = f(re^{i\theta})$  and  $1 \leq p < \infty$ , then

$$\varphi_r \rightarrow \varphi \text{ in } H^p(\mathbb{T}; F). \quad (3.3)$$

**Remark 3.5.** Motivated by the last theorem, from now on if  $\varphi \in H^p(\mathbb{T}; F)$  and  $f$  is the Cauchy integral of  $\varphi$ , we will write  $\tilde{f}$  instead of  $\varphi$ .

**Definition 3.6.** Following Bukhvalov [5] we say that a Banach space  $F$  has the analytic Radon-Nikodym property (ARNP for short) if for each  $1 \leq p \leq \infty$ , every  $f \in H^p(\mathbb{D}; F)$  satisfies (3.2). Thus, if  $F$  has ARNP and  $1 \leq p \leq \infty$ , then the mapping  $\tilde{f} \in H^p(\mathbb{T}; F) \rightarrow f \in H^p(\mathbb{D}; F)$  is an isometric isomorphism. We say  $\tilde{f}$  is the boundary value function associated with  $f$ .

We recall that a Banach space  $F$  has the Radon-Nikodym property (RNP for short) if for each finite measure space  $(X, \Sigma, \mu)$  and each measure  $\nu: \Sigma \rightarrow F$  of finite variation and absolutely continuous with respect to  $\mu$ , there exists  $f \in L^1(X, \Sigma, \mu; F)$  such that  $\nu(A) = \int_A f d\mu$  for every  $A \in \Sigma$ . Every reflexive Banach space has the RNP. We refer to the book of Diestel and Uhl [10] for background information on the RNP.

By Ryan's result every separable and reflexive Banach space has the ARNP. Bukhvalov [5] has proved that every dual Banach space with the RNP has also the ARNP. Bukhvalov and Danilevich [6] have proved that a Banach space  $F$  has the ARNP if for some  $1 \leq p \leq \infty$ , every  $f \in H^p(\mathbb{D}; F)$  satisfies (3.2). They also have proved that every Banach space with the RNP has also the ARNP. Burkholder [9] has shown that  $L^1(\mathbb{T})$  has the ARNP, but it does not have the RNP.

**Theorem 3.7.** Let  $F$  be a Banach space with the ARNP. Then  $H^p(\mathbb{D}, F)$ , for  $1 \leq p < \infty$ , is a Banach space of type  $\mathcal{H}(\mathbb{D}; F)_7$ .

**Proof:** It is enough to prove that  $P_6$  and  $P_7$  are satisfied. Property  $P_5$  follows then from Proposition 2.4 and Proposition 2.7.

Property  $P_7$  follows from equation (2.4), i.e.,  $\|T_r f\|_p = \|f\|_{p,r}$ .

By definition, the boundary value related, from the isomorphism of Theorem 3.4, to  $T_r f$  is  $f_r$ , where  $f_r(e^{i\theta}) = f(re^{i\theta})$ . From (3.3), we have that

$$\lim_{r \rightarrow 1} \|f_r - \tilde{f}\|_p = 0$$

and again from the isomorphism of Theorem 3.4,

$$\|f_r - \tilde{f}\|_p = \|T_r f - f\|_p$$

since the boundary value of  $T_r f - f$  is  $f_r - \tilde{f}$  for all  $0 \leq r < 1$ . Therefore,  $P_6$  is satisfied. ■

**Theorem 3.8.** If  $F$  has the ARNP and  $1 \leq p < \infty$ , then  $H^p(\mathbb{D}; F)'$  is isometrically isomorphic to  $H^p(\mathbb{D}; F)^b$ .

**Proof:** It follows from Propositions 2.18 and 3.7. ■

## 4 Duality and the UMD-property (UMDp)

In this section  $p, q \in (1, \infty)$  denote conjugate indices. In Proposition 2.21 we saw that  $H^q(\mathbb{D}; F') \subset H^p(\mathbb{D}; F)^b$  and we wonder when

$$H^p(\mathbb{D}; F)^b \subset H^q(\mathbb{D}; F'), \tag{4.1}$$

that is,  $H^p(\mathbb{D}; F)^b = H^q(\mathbb{D}; F')$ . The answer is not always positive. In the case of a positive answer, we prove that  $H^q(\mathbb{D}; F')$  and  $H^p(\mathbb{D}; F)'$  are topologically isomorphic spaces by Theorem 3.8. Let us express the inclusion (4.1) as an assertion:

**A<sub>1</sub>(q; F'):** If  $g \in H^p(\mathbb{D}; F)^b$ , then  $g \in H^q(\mathbb{D}; F')$ .

**Proposition 4.1.** Let  $1 < q < \infty$ . If  $\mathbf{A}_1(q; F')$  is true, then there exists a constant  $C_1(q) > 0$ , depending only on  $q$ , satisfying the inequality

$$\|g\|_{q,r} \leq C_1(q)N_q(g; r), \quad (4.2)$$

for every  $0 \leq r < 1$  and  $g \in \mathcal{H}(\mathbb{D}; F')$ . In particular, the inclusion given in (4.1) is continuous.

**Proof:** The proof in the scalar-valued case applies. See Taylor [25, Theorem 17.1]. ■

Consider the assertion:

**$\mathbf{A}_2(p; F)$ :** Let  $f \in \mathcal{H}(\mathbb{D}; F)$  be the Cauchy integral of  $\varphi \in L^p(\mathbb{T}; F)$ . Then  $f \in H^p(\mathbb{D}; F)$ .

**Proposition 4.2.** If  $\mathbf{A}_2(p; F)$  is true, then there exists a constant  $C_2(p) > 0$ , depending only on  $p$ , such that:

$$\|f\|_p \leq C_2(p)\|\varphi\|_p,$$

where  $\varphi \in L^p(\mathbb{T}; F)$  and  $f \in H^p(\mathbb{D}; F)$  is the Cauchy integral of  $\varphi$ .

**Proof:** The proof in the scalar-valued case applies. See Taylor [25, Theorem 17.2]. ■

We will now establish some implications between the assertions  $\mathbf{A}_1(q; F')$  and  $\mathbf{A}_2(p; F)$ . These assertions will be useful in Theorem 4.13 when we characterize the dual of  $H^p(\mathbb{D}; F)$  under certain conditions on  $F$ .

**Proposition 4.3.** Let  $\tilde{g} \in L^q(T; F')$ ,  $1 \leq p \leq \infty$  and  $0 \leq r < 1$ . If  $g$  is the Cauchy Integral of  $\tilde{g}$ , then  $g \in H^p(\mathbb{D}; F)^b$  and  $\|g\|_q^b \leq \|\tilde{g}\|_q$ .

**Proof:** From the fact that the Bochner integral commutes with linear functionals (see Mujica [14, page 42]), the proof given in the scalar-valued case applies. See Taylor [25, page 32, Theorem 14.1]. ■

**Proposition 4.4.**  $\mathbf{A}_1(q; F')$  implies  $\mathbf{A}_2(q; F')$ .

**Proof:** Let  $\varphi \in L^q(T; F')$  and  $g$  be the Cauchy integral of  $\varphi$ . By Proposition 4.3,  $g \in H^p(\mathbb{D}; F)^b$ . Since  $\mathbf{A}_1(q; F')$  is true,  $g \in H^q(\mathbb{D}; F')$ . ■

**Proposition 4.5.**  $\mathbf{A}_2(p; F)$  implies  $\mathbf{A}_1(q; F')$ .

**Proof:** Let  $g \in H^p(\mathbb{D}; F)^b$ ,  $\varphi \in L^p(\mathbb{T}; F)$  and  $f$  be the the Cauchy integral of  $\varphi$ . In this way, since  $f$  is the Cauchy integral of  $\varphi$ :

$$\frac{1}{2\pi} \int_0^{2\pi} \langle \varphi(e^{i\theta}), g(re^{-i\theta}) \rangle d\theta = \sum_{n=0}^{\infty} \langle c_n(\varphi), \gamma_n(g) \rangle r^n = B(f, g; r),$$

where  $c_n(\varphi) = \frac{1}{2\pi} \int_0^{2\pi} \varphi(e^{i\theta})e^{-in\theta}d\theta$ , for all  $n \in \mathbb{Z}$ . By Proposition 4.2:

$$\left| \frac{1}{2\pi} \int_0^{2\pi} \langle \varphi(e^{i\theta}), g(re^{-i\theta}) \rangle d\theta \right| \leq \|f\|_p \|g\|_q^b \leq C_2(p)\|\varphi\|_p \|g\|_q^b,$$

By a result of Diestel and Uhl [10, page 97],

$$\|g\|_{q,r} \leq C_2(p) \|g\|_q^b$$

and the proof is complete. ■

The unconditional martingale difference property (UMDP for short) has been extensively studied by Burkholder [9]. We say that  $F$  has the UMDP or that  $F$  is UMD.

Instead of giving the original definition of UMDP, for which it would be necessary to introduce several other concepts, we will give a characterization. First, we need the next definition:

**Definition 4.6.** If  $f \in L^p(\mathbb{T}; F)$ , the *analytic projection*  $f^a$  of  $f$  is the function whose negative Fourier coefficients are zero and the other coincide with the respective Fourier coefficients of  $f$ .

It is known the Hilbert transform with values in  $F$  is bounded in  $L^p(\mathbb{T}; F)$  if and only if the analytic projection is bounded. This can be proved using Hoffman [12, page 151], Riesz [19] and Burkholder [7, pages 408 and 409]. It is also known that the boundedness of Hilbert transform as an operator on  $L^p(\mathbb{T}; F)$  is equivalent to  $F$  being UMD. The sufficiency was established by Burkholder [7, 8] and the necessity by Bourgain [4]. So, we have the following theorem:

**Theorem 4.7.** A Banach space  $F$  is UMD if and only if

$$\begin{array}{ccc} S: L^p(\mathbb{T}; F) & \longrightarrow & H^p(\mathbb{T}; F) \\ f & \longmapsto & f^a \end{array}$$

is a bounded linear map for every  $1 < p < \infty$ , where  $f^a$  is the analytic projection of  $f$ .

**Remark 4.8.** An interesting result is that any UMD-space is reflexive, in fact superreflexive (see Aldous [1] or Maurey [13]). In particular, if  $F$  is UMD, then  $F$  and  $F'$  have the RNP. On the other hand Pisier [18] has constructed an example showing that a superreflexive space need not be UMD.

**Proposition 4.9.** A Banach space  $F$  with the ARNP is UMD if and only if it satisfies  $\mathbf{A}_2(p; F)$ .

**Proof:** In this proof we use the characterization of UMD-property given by Theorem 4.7. Suppose  $\mathbf{A}_2(p; F)$  is true. Then if  $\varphi \in L^p(\mathbb{T}; F)$ , the Cauchy integral  $f$  of  $\varphi$  is an element of  $H^p(\mathbb{D}; F)$  and by Bukhvalov [5, Theorem 2.3],  $\varphi$  is the unique element of  $L^p(\mathbb{T}; F)$  such that  $f$  is the Cauchy integral. By Theorem 3.4, there is only one  $\tilde{f} \in H^p(\mathbb{T}; F) \subset L^p(\mathbb{T}; F)$  where  $f$  is the Cauchy integral of  $\tilde{f}$ . It is not difficult to see that  $\tilde{f}$  is the analytic projection of  $\varphi$ . Thus,  $F$  is UMD by Proposition 4.2.

Now suppose  $F$  is UMD, that is, by Theorem 4.7 the linear map:

$$\begin{array}{ccc} S: L^p(\mathbb{T}; F) & \longrightarrow & H^p(\mathbb{T}; F) \\ \varphi & \longmapsto & \varphi^a \end{array},$$

is bounded for all  $1 < p < \infty$ , i.e., there is  $C > 0$  such that  $\|\varphi^a\|_p \leq C\|\varphi\|_p$ . If  $f$  is the Cauchy integral of  $\varphi$ , for  $z \in \mathbb{D}$  we have:

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{\varphi(e^{i\theta})}{1 - ze^{i\theta}} d\theta = \frac{1}{2\pi} \int_0^{2\pi} \frac{\varphi^a(e^{i\theta})}{1 - ze^{i\theta}} d\theta$$

and by Theorem 3.4,  $\|f\|_p = \|\varphi^a\|_p$ . This concludes the proof. ■

**Proposition 4.10.** Let  $F$  be a Banach space that has the ARNP. Then,  $\mathbf{A}_1(q; F')$  implies that  $F$  is UMD. Furthermore, if  $F'$  also has the ARNP, the converse implication is true.

**Proof:** If  $F$  is UMD, then  $\mathbf{A}_1(q; F')$  is true by Proposition 4.5 and Proposition 4.9. Now, if  $\mathbf{A}_1(q; F')$  is true, by Proposition 4.4  $\mathbf{A}_2(q; F')$  is true, and so  $F'$  is UMD by Proposition 4.9. By results of Rubio de Francia [20, page 205] and Burkholder [9, page 237], we obtain that  $F$  is UMD. ■

**Proposition 4.11.** If  $F$  has the ARNP and  $\mathbf{A}_1(q; F')$  is true, then the spaces  $H^p(\mathbb{D}; F)'$  and  $H^q(\mathbb{D}; F')$  are topologically isomorphic.

**Proof:** If  $g \in H^q(\mathbb{D}; F')$ , then from the inequalities (2.9) and (4.2), it follows that:

$$\|g\|_q^b \leq \|g\|_q \leq C_1(q) \|g\|_q^b.$$

The conclusion follows by Theorem 3.8. ■

**Definition 4.12.** We say that  $H^p(\mathbb{D}; F)'$  and  $H^q(\mathbb{D}; F')$  are canonically topologically isomorphic if  $\mathbf{A}_1(q; F')$  is true.

**Theorem 4.13.** Let  $F$  be a Banach space with ARNP such that  $F'$  also has the ARNP. Then  $H^p(\mathbb{D}; F)'$  and  $H^q(\mathbb{D}; F')$  are canonically topologically isomorphic if and only if  $F$  is UMD. In this case, we have the isomorphism

$$\begin{aligned} \Psi_p: \quad H^q(\mathbb{D}; F') &\longrightarrow H^p(\mathbb{D}; F)' \\ g &\longmapsto \Psi_p(g): \quad H^p(\mathbb{D}; F) \longrightarrow \mathbb{C} \\ f &\longmapsto \Psi_p(g)(f) = \frac{1}{2\pi} \int_0^{2\pi} \langle \tilde{f}(e^{i\theta}), \tilde{g}(e^{-i\theta}) \rangle d\theta, \end{aligned}$$

where  $\tilde{f}$  and  $\tilde{g}$  are the boundary values of  $f \in H^p(\mathbb{D}; F)$  and  $g \in H^q(\mathbb{D}; F')$ , respectively.

**Proof:** The theorem follows from Propositions 4.10 and 4.11 with the aid of Theorems 2.19 and 3.7. ■

**Remark 4.14.** For similar results see Bukhvalov [5, Theorem 3.1] and Blasco [3].

**Corollary 4.15.** If  $F$  is UMD, then  $H^p(\mathbb{D}; F)$  is reflexive when  $1 < p < \infty$ .

**Proof:** We consider the embedding  $J: H^p(\mathbb{D}; F) \longrightarrow H^p(\mathbb{D}; F)''$  and, since  $F$  is reflexive (see Remark 4.8), the isomorphisms we have obtained in Theorem 4.13:

$$\Psi_p: H^q(\mathbb{D}; F') \longrightarrow H^p(\mathbb{D}; F)' \text{ and } \Psi_q: H^p(\mathbb{D}; F) \longrightarrow H^q(\mathbb{D}; F')'.$$

Given  $x'' \in H^p(\mathbb{D}; F)''$ , we have  $x'' \circ \Psi_p \in H^q(\mathbb{D}; F')'$ . Thus, there is  $f \in H^p(\mathbb{D}; F)$  such that  $\Psi_q(f) = x'' \circ \Psi_p$ . Also, if  $x' \in H^p(\mathbb{D}; F)'$  there is  $g \in H^q(\mathbb{D}; F')$  such that  $\Psi_p(g) = x'$ . Then, it follows that

$$x''(x') = x''(\Psi_p(g)) = \Psi_q(f)(g) = \frac{1}{2\pi} \int_0^{2\pi} \langle \tilde{f}(e^{i\theta}), \tilde{g}(e^{-i\theta}) \rangle d\theta = \Psi_p(g)(f) = x'(f) = J(f)(x')$$

and the proof is complete. ■

## 5 Weak spaces of vector-valued functions

If  $A$  is a subset of some Banach space  $E$  and  $\mathcal{F}(A; F)$  is a set of functions  $f: A \rightarrow F$  satisfying some property, then the weak version of  $\mathcal{F}(A; F)$  is given by

$$\mathcal{F}(A; F)_w = \{f: A \rightarrow F; \psi \circ f \in \mathcal{F}(A; \mathbb{C}), \forall \psi \in F'\}.$$

As in the case of Hardy and Lebesgue spaces, when  $F = \mathbb{C}$  we write  $\mathcal{F}(A)$  instead of  $\mathcal{F}(A; \mathbb{C})$ .

In this section we study the weak versions of the spaces  $L^p(\mathbb{T}; F)$  and  $H^p(\mathbb{D}; F)$ :  $L^p(\mathbb{T}; F)_w$  and  $H^p(\mathbb{D}; F)_w$ , respectively. We will see some properties of these spaces. In the case of  $H^p(\mathbb{D}; F)$ , we relate the study with duality. The inspiration for this section came from a result about holomorphic functions that we can see in Mujica [14, page 65]: for all Banach space  $F$ ,  $\mathcal{H}(\mathbb{D}; F) = \mathcal{H}(\mathbb{D}; F)_w$ .

**Proposition 5.1.** For each  $1 \leq p < \infty$  and  $f \in L^p(\mathbb{T}; F)_w$  the following supremum is finite:

$$\sup_{\psi \in F', \|\psi\| \leq 1} \|\psi \circ f\|_p.$$

**Proof:** We prove the case  $1 \leq p < \infty$ . For  $p = \infty$ , the proof is analogous. For each  $f \in L^p(\mathbb{T}; F)_w$ , consider the linear map given by:

$$\begin{aligned} S_f: F' &\longrightarrow L^p(\mathbb{T}) \\ \psi &\longmapsto \psi \circ f \end{aligned}$$

We prove that  $S_f$  is continuous using the Closed Graph Theorem. From this, the result follows. In fact:

$$\|S_f\| = \sup_{\psi \in F', \|\psi\| \leq 1} \left( \frac{1}{2\pi} \int_0^{2\pi} |\psi \circ f(e^{i\theta})|^p d\theta \right)^{\frac{1}{p}}.$$

Let  $(\psi_n)_{n \in \mathbb{N}} \subset F'$  be such that  $\psi_n \rightarrow \psi \in F'$ . Now suppose  $S_f(\psi_n) \rightarrow g \in L^p(\mathbb{T})$ , that is,

$$\lim_{n \rightarrow \infty} \left( \frac{1}{2\pi} \int_0^{2\pi} \|\psi_n \circ f(e^{i\theta}) - g(e^{i\theta})\|^p d\theta \right)^{\frac{1}{p}} = 0.$$

By Bartle [2, page 73], there exists a subsequence  $(\psi_{n_k} \circ f)$  such that  $\psi_{n_k} \circ f(e^{i\theta}) \rightarrow g(e^{i\theta})$  almost everywhere. Thus, also almost everywhere

$$g(e^{i\theta}) = \psi \circ f(e^{i\theta}),$$

that is,  $g = S_f(\psi) \in L^p(\mathbb{T})$ . ■

**Definition 5.2.** For each  $f \in L^p(\mathbb{T}; F)_w$  and  $1 \leq p \leq \infty$ , we write

$$\|f\|_p^w = \sup_{\psi \in F', \|\psi\| \leq 1} \|\psi \circ f\|_p.$$

**Proposition 5.3.** Let  $1 \leq p \leq \infty$ , let  $F$  be a Banach space and suppose that there exists a sequence  $(\psi_n) \subset F'$  which separates the points of  $F$ . Then:

- a)  $(L^p(\mathbb{T}; F)_w, \|\cdot\|_p^w)$  is a normed space.
- b)  $(L^p(\mathbb{T}; F)_w, \|\cdot\|_p^w)$  is isometrically isomorphic to a subspace of  $\mathcal{L}(F'; L^p(\mathbb{T}))$ .
- c)  $(L^p(\mathbb{T}; F)_w, \|\cdot\|_p^w)$  is a Banach space.

**Proof:** It is easy to see that  $(L^p(\mathbb{T}; F); \|\cdot\|_p^w)$  is a seminormed space for every  $F$ . If  $f \in L^p(\mathbb{T}; F)$  and  $\|f\|_p^w = 0$ , then it follows that  $\psi_n \circ f = 0$  a.e. for every  $n \in \mathbb{N}$ . Since  $(\psi_n)$  separates the points of  $F$ , it follows that  $f = 0$  a.e.

For assertion (b), define the map

$$\begin{array}{ccc} S: L^p(\mathbb{T}; F)_w & \longrightarrow & \mathcal{L}(F'; L^p(\mathbb{T})) \\ f & \longmapsto & S_f, \end{array}$$

where  $S_f$  was defined in the proof of Proposition 5.1. Of course  $S$  is well-defined and is an isometric map:

$$\|S_f\| = \sup_{\psi \in F', \|\psi\| \leq 1} \|S_f(\psi)\| = \sup_{\psi \in F', \|\psi\| \leq 1} \|\psi \circ f\|_p = \|f\|_p^w.$$

Also,  $S$  is injective and linear. Assertion (c) is a consequence of assertion (b) and of the fact that  $\mathcal{L}(F'; L^p(\mathbb{T}))$  is a Banach space.

**Remark 5.4.** There exists a sequence  $(\psi_n) \subset F'$  which separates the points of  $F$  if and only if there exists an injective operator  $T \in L(F; l_\infty)$ . In particular every separable Banach space verifies this condition.

**Theorem 5.5.** For every Banach space  $F$  the following conditions are equivalent.

- a)  $L^p(\mathbb{T}; F) = L^p(\mathbb{T}; F)_w$  for all  $1 \leq p < \infty$ .
- b)  $L^p(\mathbb{T}; F) = L^p(\mathbb{T}; F)_w$  for some  $1 \leq p < \infty$ .
- c)  $F$  has finite dimension.

**Proof:** Of course (a)  $\implies$  (b). For (b)  $\implies$  (c), consider a sequence  $(x_n)_{n \in \mathbb{N}} \subset F$  such that

$$\sum_{n=1}^{\infty} |\varphi(x_n)|^p < \infty$$

for all  $\varphi \in F'$  and  $1 \leq p < \infty$ . Our aim is to show that  $\sum_{n=1}^{\infty} \|x_n\|^p < \infty$ , i.e., to prove that the identity  $I: F \rightarrow F$  is absolutely  $p$ -summing. From this, by Pietsch [17],  $F$  has finite-dimension.

Define

$$f(e^{i\theta}) = \sum_{n=1}^{\infty} \frac{x_n}{2^{\frac{-(n+1)}{p}} \pi^{\frac{-1}{p}}} \chi_{I_n}(e^{i\theta}),$$

where  $0 \leq \theta \leq 2\pi$ ,  $\{I_n\}_{n=1}^{\infty}$  is a sequence of disjoint intervals,  $\chi_{I_n}$  is the characteristic function of  $I_n$  and  $\frac{1}{2\pi} \int_{I_n} d\theta = \frac{1}{2^n}$ . We assert that  $f \in L^p(\mathbb{T}; F)_w$ . Indeed, it is plain that for each  $\varphi \in F'$ ,  $\varphi \circ \tilde{f}$  is measurable and

$$\left( \frac{1}{2\pi} \int_0^{2\pi} |\varphi(f(e^{i\theta}))|^p d\theta \right)^{\frac{1}{p}} = \left( \sum_{n=1}^{\infty} \frac{1}{2\pi} \int_{I_n} |\varphi(f(e^{i\theta}))|^p d\theta \right) = \left( \sum_{n=1}^{\infty} |\varphi(x_n)|^p \right)^{\frac{1}{p}} < \infty.$$

By hypothesis,  $f \in L^p(\mathbb{T}; F)$  and

$$\left( \sum_{n=1}^{\infty} \|x_n\|^p \right)^{\frac{1}{p}} = \left( \sum_{n=1}^{\infty} \frac{1}{2\pi} \int_{I_n} \|f(e^{i\theta})\|^p d\theta \right)^{\frac{1}{p}} = \left( \frac{1}{2\pi} \int_0^{2\pi} \|f(e^{i\theta})\|^p d\theta \right)^{\frac{1}{p}} < \infty.$$

Now, let us prove that (c)  $\implies$  (a). We just have to show  $L^p(\mathbb{T}; F)_w \subset L^p(\mathbb{T}; F)$ . The other inclusion is obvious for all  $F$ . Let  $\{e_1, \dots, e_n\}$  be a basis of  $F$  with dual basis  $\{\varphi_1, \dots, \varphi_n\}$ . Let  $f \in L^p(\mathbb{T}; F)_w$  and write  $f = f_1 e_1 + \dots + f_n e_n$ , where  $f_j = \varphi_j \circ f$  for  $j = 1, \dots, n$ . Of course each  $f_j \in L^p(\mathbb{T})$ . Furthermore,

$$\left( \frac{1}{2\pi} \int_0^{2\pi} |f_j(e^{i\theta})|^p d\theta \right)^{\frac{1}{p}} = \left( \frac{1}{2\pi} \int_0^{2\pi} |\varphi_j(f(e^{i\theta}))|^p d\theta \right)^{\frac{1}{p}} < \infty$$

resulting that  $f \in L^p(\mathbb{T}; F)$ . ■

The proof of this theorem follows a suggestion of Professor Oscar Blasco.

**Proposition 5.6.** If  $1 \leq p \leq \infty$  and  $f \in H^p(\mathbb{D}; F)_w$ , the following supremum is finite:

$$\sup_{\psi \in F', \|\psi\| \leq 1} \|\psi \circ f\|_p.$$

**Proof:** If  $f \in H^p(\mathbb{D}; F)_w$ , define

$$\begin{aligned} \Xi_f: F' &\longrightarrow H^p(\mathbb{D}) \\ \psi &\longmapsto \psi \circ f \end{aligned}.$$

Let  $(\psi_n)_{n \in \mathbb{N}} \subset F'$  such that  $\psi_n \rightarrow \psi \in F'$ . Suppose  $\Xi_f(\psi_n) \rightarrow g \in H^p(\mathbb{D})$  i.e.,



$$\lim_{n \rightarrow \infty} \sup_{0 \leq r < 1} \left( \frac{1}{2\pi} \int_0^{2\pi} \|\psi_n \circ f(re^{i\theta}) - g(re^{i\theta})\|^p d\theta \right)^{\frac{1}{p}} = 0.$$

So, for each  $r$  we have

$$\lim_{n \rightarrow \infty} \left( \frac{1}{2\pi} \int_0^{2\pi} \|\psi_n \circ f(re^{i\theta}) - g(re^{i\theta})\|^p d\theta \right)^{\frac{1}{p}} = 0$$

and proceeding as in the proof of Proposition 5.1, we obtain almost everywhere

$$g(re^{i\theta}) = \psi \circ f(re^{i\theta}). \quad (5.1)$$

Since  $g, f$  e  $\psi$  are continuous (in particular,  $f \in \mathcal{H}(\mathbb{D}; F)$ ), for each  $0 \leq r < 1$ ,  $g(re^{i\theta}) = \psi \circ f(re^{i\theta})$  for all  $0 \leq \theta \leq 2\pi$ . Thus  $g = \psi \circ f$ . By the Closed Graph Theorem,  $\Xi_f$  is continuous and the result follows. For  $p = \infty$ , the proof is analogous and we can use Lemma 2.6. ■

**Definition 5.7.** For each  $f \in H^p(\mathbb{D}; F)_w$  and  $1 \leq p \leq \infty$ , we write

$$\|f\|_p^w = \sup_{\psi \in F', \|\psi\| \leq 1} \|\psi \circ f\|_p.$$

**Proposition 5.8.** Let  $1 \leq p \leq \infty$ . Then:

- a)  $(H^p(\mathbb{D}; F)_w, \|\cdot\|_p^w)$  is a Banach space of type  $\mathcal{H}(\mathbb{D}; F)_4$  and  $(H^\infty(\mathbb{D}; F)_w, \|\cdot\|_\infty^w) = (H^\infty(\mathbb{D}; F), \|\cdot\|_\infty)$ .
- b) If  $F$  is reflexive, then  $(H^p(\mathbb{D}; F)_w, \|\cdot\|_p^w)$  is isometrically isomorphic to  $\mathcal{L}(F'; H^p(\mathbb{D}))$ .

**Proof:** If  $f \in H^p(\mathbb{D}; F)_w$  and  $\|f\|_p^w = 0$ , note that  $\psi \circ f = 0$  a.e. for all  $\psi \in F'$ . Since  $f$  and  $\psi$  are continuous, then  $\psi \circ f \equiv 0$ , for all  $\psi \in F'$ , which implies  $f \equiv 0$ . The other properties of norm are easy to prove.

To show the fact that is a Banach space of type  $\mathcal{H}(\mathbb{D}; F)_4$  we use a consequence of the Hahn-Banach Theorem, that asserts for every  $v \in F$  that

$$\|v\| = \sup_{\|\psi\| \leq 1, \psi \in F'} |\psi(v)|,$$

and the steps of the proof of Taylor [25, Theorem 11.1]. See also Proposition 2.7. The second part of (a) is a consequence of the Banach-Steinhaus Theorem.

For assertion (b), define the map

$$\begin{array}{ccc} \Xi: H^p(T; F)_w & \longrightarrow & \mathcal{L}(F'; H^p(\mathbb{D})) \\ f & \longmapsto & \Xi_f, \end{array}$$

where  $\Xi_f$  was defined in Proposition 5.6. Of course  $\Xi$  is well-defined and is an isometric map:

$$\|\Xi_f\| = \sup_{\psi \in F', \|\psi\| \leq 1} \|\Xi_f(\psi)\| = \sup_{\psi \in F', \|\psi\| \leq 1} \|\psi \circ f\|_p = \|f\|_p^w.$$

Also,  $\Xi$  is injective and linear. We just need to show  $\Xi$  is surjective. Let  $R \in \mathcal{L}(F'; H^p(\mathbb{D}))$ . Then, for each  $0 \leq r < 1$  and  $e^{i\theta} \in T$ , we define the following element of  $F''$ :

$$\begin{aligned} R_{\theta,r}: F' &\longrightarrow \mathbb{C} \\ \psi &\longmapsto R(\psi)(re^{i\theta}) \end{aligned}$$

Indeed,  $R_{\theta,r}$  is linear and also continuous, since is the composition of the mapping  $R$ , which is continuous, and of the evaluation at the point  $re^{i\theta}$ , which is continuous by Proposition 2.10.

Since  $F$  is reflexive, there exists a unique  $a_{\theta,r} \in F$  such that  $R_{\theta,r}(\psi) = \psi(a_{\theta,r})$ , for all  $\psi \in F'$ . Then is well-defined the function

$$\begin{aligned} f: \mathbb{D} &\longrightarrow F \\ re^{i\theta} &\longmapsto a_{\theta,r} \end{aligned}$$

and for all  $0 \leq r \leq 1$ ,  $e^{i\theta} \in \mathbb{T}$  and  $\psi \in F'$  it follows that

$$\psi(f(re^{i\theta})) = \psi(a_{\theta,r}) = R_{\theta,r}(\psi) = R(\psi)(re^{i\theta})$$

which implies that  $\psi \circ f = R(\psi)$ , resulting that  $f \in H^p(\mathbb{D}; F)_w$  and  $\Xi_f = R$ . ■

**Theorem 5.9.** Let  $p, q \in (1, \infty)$  be conjugate indices, and let  $F$  be a Banach space such that  $F$  and  $F'$  have the ARNP. If  $H^p(\mathbb{D}; F)_w = H^p(\mathbb{D}; F)$ , then:

- a)  $H^p(\mathbb{D}; F)'$  and  $H^q(\mathbb{D}; F')$  are canonically topologically isomorphic.
- b)  $F$  is UMD.

**Proof:** Let us show that the equality  $H^p(\mathbb{D}; F)_w = H^p(\mathbb{D}; F)$  implies  $\mathbf{A}_2(p; F)$ . The rest of the proof follows from Propositions 4.5 and 4.11 and Theorem 4.13. Let  $\varphi \in L^p(\mathbb{T}; F)$  and let  $f \in \mathcal{H}(\mathbb{D}; F)$  be its Cauchy integral. For all  $\psi \in F'$ , we have  $\psi \circ \varphi \in L^p(\mathbb{T})$  and by Taylor [25, page 45] we obtain  $\psi \circ f \in H^p(\mathbb{D})$ . By hypothesis,  $f \in H^p(\mathbb{D}; F)$ . It concludes the proof. ■

**Example 5.10.** Let  $p, q \in (1, \infty)$  be conjugate indices. If  $F = H^q(\mathbb{D})$ , then  $F$  is UMD, but  $H^p(\mathbb{D}; F)_w \neq H^p(\mathbb{D}; F)$ .

Suppose that  $H^p(\mathbb{D}; F)_w = H^p(\mathbb{D}; F)$  for all  $1 < p < \infty$ . We know that  $H^q(\mathbb{D})$  and  $H^q(\mathbb{T})$  are isometrically isomorphic. By Pelczynski [16, page 8],  $H^q(\mathbb{T})$  and  $L^q(\mathbb{T})$  are isomorphic. By Burkholder [9, page 237], every Banach space isomorphic to a UMD-space is also UMD and  $L^q(\mathbb{T})$  is UMD. Hence, we obtain that  $H^q(\mathbb{D})$  is UMD. It follows by Proposition 5.8, Corollary 4.15, and the assumption  $H^p(\mathbb{D}; F)_w = H^p(\mathbb{D}; F)$  that  $\mathcal{L}(F'; H^p(\mathbb{D}))$  is reflexive. Let  $\mathcal{L}_k(F'; H^p(\mathbb{D}))$  denote the subspace of all  $T \in \mathcal{L}(F'; H^p(\mathbb{D}))$  which are compact. Also, since  $L^q(\mathbb{T})$  has the approximation property (see Diestel and Uhl [10, page 245]), it follows that  $H^q(\mathbb{D})$  has the approximation property and by Mujica [15, Theorem 2.1] we have that  $\mathcal{L}(F'; H^p(\mathbb{D})) = \mathcal{L}_k(F'; H^p(\mathbb{D}))$ . So, if we consider  $F = H^q(\mathbb{D})$  it follows that  $F' = H^p(\mathbb{D})$  and then

$$\mathcal{L}(H^p(\mathbb{D}); H^p(\mathbb{D})) = \mathcal{L}_k(H^p(\mathbb{D}); H^p(\mathbb{D})).$$

This implies that the identity  $I: H^p(\mathbb{D}) \longrightarrow H^p(\mathbb{D})$  is compact, which is false, since  $H^p(\mathbb{D})$  is a infinite-dimensional space. ■

An example where the equality  $H^p(\mathbb{D}; F)_w = H^p(\mathbb{D}; F)$  occurs is when  $F$  is finite dimensional. We have just seen an example where the equality  $H^p(\mathbb{D}; F)_w = H^p(\mathbb{D}; F)$  is not true, where  $F$  is an infinite dimensional space.

**Conjecture 5.11.** We conjecture that  $H^p(\mathbb{D}; F) = H^p(\mathbb{D}; F)_w$  if and only if  $F$  is finite dimensional.

## References

- [1] D. J. Aldous, *Unconditional bases and martingales in  $L_p(F)$* , Math. Proc. Cambridge Philos. Soc. (1979), 117-123.
- [2] R. G. Bartle, *The Elements of Integration and Lebesgue Measure*, New York, Wiley Classics Library, 1995.
- [3] O. Blasco, *Espacios de Hardy de Funciones con Valores Vectoriales*, Tesis, Zaragoza, 1985.
- [4] J. Bourgain, *Some remarks on Banach spaces in which martingale difference sequences are unconditional*, Ark. Mat. 21 (1983), 163-168.
- [5] A. V. Bukhvalov, *Hardy spaces of vector-valued functions*, J. Sov. Math. 16(1981), 1051-1059.
- [6] A. V. Bukhvalov e A. A. Danilevich, *Boundary properties of analytic and harmonic functions with values in Banach space*, Mat. Zametki 31 (1982), 203-214 [Russian]; Math. Notes Acad. Sci. USSR 31 (1982), 104-110.
- [7] D. L. Burkholder, *A geometric condition that implies the existence of certain singular integrals of Banach-space-valued functions*, Proc. Conf. Harmonic Analysis in Honour of A. Zygmund, Univ. of Chicago, 1981, Wadsworth International Group, Belmont, California, 1 (1983), 270-286.
- [8] D. L. Burkholder, *Martingales and Fourier analysis in Banach spaces*, Lecture Notes in Math., vol. 1206, Springer-Verlag, Berlin, Heilberg and New York, 1986, 61-108.
- [9] D. L. Burkholder, *Martingales and singular integrals in Banach spaces*, in: Handbook of the Geometry of Banach Spaces, Vol. I, 233-269, North-Holland, Amsterdam, 2001.
- [10] J. Diestel e J.J.Uhl, *Vector Measures*, Amer. Math. Soc., 1977.
- [11] P. L. Duren, *Theory of  $H^p$  Spaces*, Academic Press, New York, 1970.
- [12] K. Hoffman, *Banach Spaces of Analytic Functions*, Prentice-Hall, Englewood Cliffs, NJ, 1962.
- [13] B. Maurey, *Système de Haar*, Sminaire Maurey-Schwartz, 1974-1975, Ecole Polytechnique, Paris, 1975.
- [14] J. Mujica, *Complex Analysis in Banach Spaces*, Math. Studies, **120**, North-Holland, Amsterdam, 1986

- [15] J. Mujica, *Reflexive spaces of homogeneous polynomials*, Bull. Polish Acad. Sci. Math. 49 (2001), 211-222.
- [16] A. Pelczynski, *Banach spaces of analytic functions and absolutely summing operators*, CBMS Regional Conf. Series 30, Amer. Math. Soc., 1977.
- [17] A. Pietsch, *Absolute  $p$ -summierende Abbildungen in normierten Raumen*, Studia Math. 28 (1967), 333-353.
- [18] G. Pisier, *Un exemple concernant la super-reflexivité*, Seminaire Maurey-Schwartz, Ecole Polytechnique, Paris, 1975.
- [19] F. Riesz, *ber die Randwerte einer analytischen Funktion*, Math. Z. 18 (1923), 87-95.
- [20] J. L. Rubio de Francia, *Martingale and integral transforms of Banach space valued functions*, Lecture Notes in Math., vol. 1221, Springer-Verlag, Berlin-New York, 1986, 195-222.
- [21] W. Rudin, *Real and Complex Analysis*, McGraw-Hill, New York, 1966.
- [22] R. Ryan, *Boundary values of analytic vector valued functions*, Indag. Math. 65 (1962), 558-572.
- [23] R. Ryan, *The  $F$ . and  $M$ . Riesz theorem for vector measures*, Indag. Math. 66 (1963), 408-412.
- [24] A. E. Taylor, *Banach spaces of functions analytic in the unit circle I*, Studia Math. 11 (1950), 145-170.
- [25] A. E. Taylor, *Banach spaces of functions analytic in the unit circle II*, Studia Math. 12 (1951), 25-50.
- [26] E. Thorp e R. Whitley, *The strong maximum modulus theorem for analytic functions into Banach spaces*, Proc. Amer. Math. Soc. 18 (1967), 640-646.